Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational Physics 227 (2008) 2130–2153
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Time behaviour of the error when simulating finite-band periodic waves. The case of the KdV equation

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Received 9 November 2006; received in revised form 18 October 2007; accepted 23 October 2007
Available online 1 November 2007

Abstract

This paper is devoted to study the error growth of numerical time integrators for N-phase or N-band quasi-periodic (in time) solutions of the periodic Korteweg–de Vries equation. It is shown that the preservation, through numerical time integration, of conserved quantities of the periodic problem of the equation, may be an element to take into account in the selection of the numerical method. We explain why the inclusion of these properties of conservation provides a better error propagation. In particular, we emphasize how the preservation of invariants makes influence in the simulation of some physical parameters of the waves.

MSC: 65M20; 65M99; 35Q53; 76B25

Keywords: KdV; Cnoidal waves; Finite-band solutions; Conserved quantities; Conservative methods

1. Introduction

This work may be considered as a contribution to extend the advantages of the so-called geometric integrators in the numerical approximation of some partial differential equations. These methods are designed taking into account geometric properties of the differential equations being integrated. Many references about the benefits of their use in ordinary differential equations can be cited ([53,27,52] and references therein). The use of geometric integrators in partial differential equations, although experimentally suitable in some cases, does not seem to be justified theoretically in a similar level to the case of ordinary differential equations. Some works in this sense are [1,4,32] and references therein.

This paper is interested in the behaviour in time of simulations of the solution of the periodic problem of the Korteweg–de Vries (KdV) equation with finite-band potentials as initial conditions. Its main conclusion is that the preservation, or approximate preservation up to certain order of the step size, of invariant quantities of the problem through the numerical time integration, is a property suitable for a numerical method, in the

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doi:10.1016/j.jcp.2007.10.016

sense that it provides a better error propagation in time than in the nonconservative case. The structure in time of the error is analyzed, leading to the identification of some sources of harmful propagation. A consequence of this analysis involves the simulation of some parameters of the wave. It is shown that a good behaviour of the method with respect to the first $N+1$ invariants of the problem gives a more correct simulation of the parameters that describe an N-band periodic solution.

The paper is structured as follows. Section 2 is devoted to explain some background material on the problem considered. This includes the introduction to the periodic Korteweg–de Vries equation, with its Hamiltonian structure and the generation of infinite conserved quantities. This Section also gives an overview about finite-band solutions of the KdV equation; some of them can be characterized in terms of the invariants of the problem. We also make a description of the parameters that define these solutions. Then we divide the study of the corresponding numerical approximation into two sections. In Section 3 we treat the case of one-band solutions, the well-known cnoidal waves. Some reasons to separate this case from the general one can be given:

- The explicit expression of the solutions allows to illustrate the results about the time behaviour of the errors and to study numerically the behaviour of the parameters in a more detailed way.
- The one-phase case may serve to treat the general case, in the sense that, due to the structure of an N-phase solution (with $N>1$), the study done for the cnoidal waves may help to obtain some conclusions about the propagation of the error in the multi-phase case.
- The analysis of the error can be given in a different and simpler manner in the one-phase case. This is due to the fact that the cnoidal waves are relative equilibria of the problem. The spectral analysis of the linearized problems around the cnoidal waves, that we need to state in order to investigate the structure of the error, can be done by using some properties that are not available, at least apparently, in the general case. The main one is the possibility of transforming the linearized equation around a one-phase solution into a constant in time coefficients equation.

Section 3 shows the main results about the time behaviour of the numerical methods when approximating cnoidal waves. It includes the application of the spectral analysis of the linearization of the KdV equation around a cnoidal wave to the numerical solutions, via asymptotic expansions of the error. This section ends with a subsection of numerical experiments. This consists of the illustration of the previous theoretical results and a study about the influence of the conservation properties in the simulation of the parameters of the wave. Finally, Section 4 treats the case of general N-band solutions, with $N>1$. From some intuitive ideas about the influence of the study of Section 3 in this general case, we then analyze the linearization around the N-band solution from a different point of view. Here we use the description of the solutions of the corresponding linearized KdV equation around a N-band potential, given by the squared Bloch eigenfunctions of the associated Hill's problem. This will allow to identify some harmful sources of the error and their relations with the parameters that characterizes the solution.

The analysis is performed for semi-discrete in time numerical methods. Other studies for fully discrete schemes can also be done [9] and may be the subject of future extensions.

2. The periodic KdV equation and finite-band solutions

We are concerned with the Korteweg–de Vries equation [33]

$$u_t + 2uu_x + u_{xxx} = 0, \quad -\infty < x < \infty, \ \ t > 0, \tag{1}$$

with periodic boundary conditions, of period say L,

$$u(x+L,t) = u(x,t). \tag{2}$$

Here, $u = u(x,t)$ is real-valued. It would be very difficult to mention even a small part of the studies involving (1) or (1) and (2) and its importance in many scientific disciplines. We may cite [23,37–39,45,47,54,58] as a modest representation. It is also of interest for us to emphasize some properties of (1) and (2) that will be used later (they are susceptible of being ordered in several ways). The first one is the integrable character of (1) and (2). The method of inverse scattering ([23,37,59,2,3,46,48] between others), originally developed for the initial

value problem of the KdV equation, can also be adapted to the periodic case [35,47]. Some consequences of this fact deserve some comments. The first one is the existence of an infinite number of functionals that are preserved in time by the solutions of (1) and (2). As in the case of the initial value problem, they can be given hierarchically by using a Lenard recursion

$$\mathbb{J}\delta H_{n+1}(u) = \mathbb{K}\delta H_n(u), \quad n = 1, 2, \ldots, \tag{3}$$

(δ denotes variational derivative) where $\mathcal{J}, \mathcal{K}$ are the operators

$$\mathbb{J} = \partial_x, \quad \mathbb{K} = -\partial_{xxx} - u\partial_x - \partial_x u.$$

The first three invariants are

$$H_0(u) = \int_0^L u \, \mathrm{d}x, \tag{4}$$

$$H_1(u) = \frac{1}{2}\int_0^L u^2 \, \mathrm{d}x, \tag{5}$$

$$H_2(u) = \int_0^L \left(\frac{1}{2}u_x^2 - \frac{1}{3}u^3\right) \mathrm{d}x. \tag{6}$$

The second quantity $H_1(u)$ is associated to the symmetry group of (1) of translations in space, with $\mathbb{J}\delta H_1(u)$ as the infinitesimal generator of the group [49]. On the other hand, the group of translations in time is the flow of the Hamiltonian vector field $\mathbb{J}\delta H_2(u)$ associated to (6). This gives the Hamiltonian form of Eq. (1)

$$u_t = \mathbb{J}\delta H_2(u). \tag{7}$$

In order to adapt the conditions (2) to the Hamiltonian structure, we will consider (7) defined in the phase space H^1_{per} of periodic, H^1 functions with period L, with the usual norm

$$||u||_1^2 = \int_0^L u^2 \, \mathrm{d}x + \int_0^L u_x^2 \, \mathrm{d}x.$$

The first invariant (4) is called a Casimir [38,40,49]. It is not related to any transformation group and it is a consequence of the degenerate character of the operator $\mathbb{J}$, whose kernel is spanned by the gradient $\delta H_0(u)$. Here we also introduce the subspace of H^1_{per} where the operator $\mathbb{J}$ is invertible,

$$H^1_{\mathrm{per},0} = \{u \in H^1_{\mathrm{per}} / H_0(u) = 0\},$$

that we will need later. It consists of L-periodic, H^1 functions with zero mean value. The rest of the invariants in the hierarchy (3) does not correspond to any geometric symmetry of (1), although some symmetries associated to the quantities can be defined in a generalized sense [49].

The inverse scattering method may also allow to construct special solutions of (1) and (2), from finite-band potentials [3,14,42,51]. The resulting solutions are conditionally periodic or quasi-periodic, that is functions of a finite number of phase variables $\theta_i = \kappa_i x - \omega_i t$, periodic in each θ_i but with, in general, noncommesurable wave numbers κ_i. They can be expressed in terms of hyperelliptic functions or by using the Theta function representation [3]. Here we will consider those waves that are periodic, that is, corresponding to commensurable wave numbers. Following Lax [38,39], for instance, the time evolution of these waves is quasi-periodic.

We make a brief description of these solutions. More details and a much better explanation can be seen in many places [3,11,46,51]. One may start from the Hill's equation (also called Schrödinger equation) [41]

$$-y'' + q(x)y = \lambda y, \quad q(x + L) = q(x). \tag{8}$$

In quantum mechanics, q plays the role of the potential of the system. The spectrum of (8) can be determined by using Floquet theory. It consists of a sequence of intervals (finite or infinite), the allowed energy or stable bands. The length of the separating gaps between the intervals (the forbidden energy bands) tends to zero when $\lambda \to \infty$. There are two linearly independent Bloch eigenfunctions. The main spectrum is composed of those eigenvalues $\lambda_i = E_i$ for which at least one of the eigenfunctions is L-periodic or $2L$-periodic. We may assume that the stable bands are the intervals $[E_{2i-1}, E_{2i}]$ while the gaps are $[E_{2i}, E_{2i+1}]$. For our purposes it

is also necessary to consider the auxiliary spectrum, which is the spectrum of the Dirichlet problem given by (8) plus the conditions $y(x_0) = 0$, $y(x_0 + T) = 0$, for an arbitrary $x_0 \in [0, T]$. This spectrum is discrete, with the eigenvalues $\gamma_i = \gamma_i(x_0)$ in the unstable bands, and only one γ_i in each $[E_{2i}, E_{2i+1}]$.

Special role in our context is played by the so-called finite-band or finite-gap potentials They are potentials q_N for which the number of intervals is finite (typically a finite number of bounded intervals and one of infinite length). Then, for an N-band potential with $N \geqslant 1$, the main spectrum contains a finite number of simple eigenvalues $E_i, i = 1, \ldots, 2N + 1$. The corresponding eigenfunction is L-periodic or $2L$-periodic (antiperiodic). The main spectrum also contains an infinite number of double eigenvalues with a two-dimensional eigenspace.

The direct scattering transform consists of the determination of the main spectrum E_j, $j = 1, \ldots, 2N + 1$, from the initial data q_N, by solving the corresponding eigenvalue problem (8). The inverse scattering transform is based on the construction of the functions γ_j at any point x and the reconstruction of the potential u from the known set of band edges E_j, $j = 1, \ldots, 2N + 1$, via the trace formula [3,11]

$$q_N = \sum_{j=1}^{2N+1} E_j - 2\sum_{j=1}^{N} \gamma_j, \tag{9}$$

and the evolution in space given by the system

$$\begin{aligned} &\frac{\mathrm{d}\gamma_j}{\mathrm{d}x} = \frac{2i\sigma_j\sqrt{R(\gamma_j)}}{\prod_{k=1,k\neq j}^{N}(\gamma_i - \gamma_k)}, \quad \sigma_j = \pm 1, \ \ j = 1, \ldots, N, \\ &\gamma(x_0) = \gamma_j^0, \quad j = 1, \ldots, N, \end{aligned} \tag{10}$$

where

$$R(E) = \sum_{i=1}^{2N+1}(E - E_i).$$

On the other hand, the E_i do not depend on x. In the context of the resolution of (1) and (2), the previous steps must be completed with the time dependence stage. This shows [3,48] that the E_i are independent of time, while the evolution of the γ_i follows a system of the form:

$$\begin{aligned} &\frac{\mathrm{d}\gamma_j}{\mathrm{d}t} = F(\gamma_1, \ldots, \gamma_N, E_1, \ldots, E_{2N+1})\sqrt{R(\gamma_j)}, \\ &\gamma_j(x, 0) = \gamma_j(x), \quad \sigma_j' = \pm 1, \ \ j = 1, \ldots, N, \end{aligned} \tag{11}$$

for some function F. Thus, from an N-band potential ($N \geqslant 1$), $u(x, 0) = q_N(x)$, we may reconstruct the solution of (1) and (2): first one determines the band edges E_i, $i = 1, \ldots, 2N + 1$, the part of the auxiliary spectrum γ_i^0, $i = 1, \ldots, N$ in the unstable bands $[E_{2i}, E_{2i+1}]$ at one point, say $x_0 = 0$, and the signs σ_j specified there. Then one solves (10) for $\gamma_i(x) = \gamma_i(x, 0)$. These form the initial conditions for (11) to get the evolution of $\gamma_i(x, t)$, $t \geqslant 0$ and, consequently, the solution $u(x, t)$ of (1) and (2) via (9) for each t. See e.g. [39] for a description in terms of the Lax pairs.

The solution (9) can also be written in terms of the so-called Riemann theta function [3,14,11]

$$u_N = -\frac{\partial^2}{\partial x^2}\ln \Theta(\eta_1, \ldots, \eta_N) + C, \tag{12}$$

with C constant, Θ the Riemann theta function and

$$\eta_j = \kappa_j x - \omega_j t + \phi_j, \quad j = 1, \ldots, N, \tag{13}$$

where κ_j are the wave numbers, ω_j the frequencies and ϕ_j the constant phases. This representation can be obtained from the previous one by using Abelian transformations and Riemann surfaces [14]. The particular case $N = 1$ gives the well-known cnoidal wave solution

$$u_1 = b_2 + (b_3 - b_2)cn^2\left(\sqrt{\frac{(b_3 - b_1)}{6}}(x - ct - x_0), m\right). \tag{14}$$

Here, $b_3 > b_2 > b_1$ are constants and $\mathrm{cn}(u,m)$ is the Jacobian elliptic cosine of modulus m. The period of the wave is

$$L = \frac{4\sqrt{6}K(m)}{\sqrt{b_3 - b_1}},$$

where $K(m)$ is the complete elliptic integral of the first kind and m is the modulus

$$m = \frac{b_3 - b_2}{b_3 - b_1}$$

with $0 < m < 1$. The wave number and the frequency of u are $\kappa = 2\pi/L$, $\omega = \kappa c$. Some other relevant parameters are: the speed $c := \frac{b_3 - b_2}{3}(2b_1 - b_2 + 2b_3)$; the amplitude $A := b_3 - b_2$; the initial phase x_0; the maximum (b_3) and the minimum (b_2). In this form, cnoidal wave profiles are determined by the three parameters b_1, b_2, b_3. They can also be represented in terms of the band edges E_1, E_2, E_3.

An alternative way to introduce the finite-band periodic solutions involves the invariants of (1) and (2) given by the recursion (3) [38,39]. Let us first consider the case of cnoidal waves. They are constrained minima of the Hamiltonian H_2 on level sets of the invariants H_1, for a given mean value. For a fixed level set $\{H_1(u) = \gamma\}$, a cnoidal wave profile u_0 satisfies

$$\begin{aligned} &\mathbb{J}(\delta H_2(u_0) + \lambda \delta H_1(u_0)) = 0, \\ &H_1(u_0) = \gamma. \end{aligned} \tag{15}$$

That is, $u_0 \in H^1_{\mathrm{per}}$ is a stationary point of H_2 restricted to the level set, or relative equilibrium. We must give some remarks.

(i) The symmetry group of translations in space determines the cnoidal wave solutions of (1) and (2), since:
- Any translation $u(x - x_0)$ of a solution u of (15) is also a solution of (15).
- The time evolution of the cnoidal wave solution, from a profile u_0 given by (15) can be obtained by applying the symmetry group of translations determined by the Lagrange multiplier λ [38], that is

$$u(x,t) = u_0(x - \lambda t - x_0). \tag{16}$$

Thus, λ plays the role of the speed of the wave.

(ii) Within this framework, the mean value of the cnoidal solution plays a relevant role. Note that (15) can be written as

$$\begin{aligned} &\delta H_2(u_0) + \lambda \delta H_1(u_0) = -\alpha \delta H_0(u_0), \\ &u_0'' + u_0^2 - \lambda u_0 = \alpha, \end{aligned} \tag{17}$$

for some constant α. By integrating in a period, we have

$$\alpha(\gamma,\mu) = \frac{1}{L}(2\gamma - \lambda(\gamma,\mu)\mu), \mu = \int_0^L u_0 \, \mathrm{d}x.$$

Also, the existence of a smooth family $(\gamma,\mu) \mapsto u_0(\gamma,\mu)$ of solutions of (17) for $\gamma > 0$, $\mu \in \mathbb{R}$ can be proved [38]. Theoretical studies about cnoidal waves (specially studies about stability, persistence, etc.) usually consider only the case of zero mean value, by using the Galilean invariance of the KdV. However, in our case mean value plays some role in the analysis and we are forced to consider it as a general parameter, since it has some relevant influence in the behaviour of the numerical approximations. Due to the dependence on γ and μ, we will denote the solution u in (16) as $u(x,t,\gamma,\mu,x_0)$.

It may be worth noting that, under certain conditions [38,34,40], a general N-phase profile also satisfies a problem like (17) but with the first $N+1$ invariants of (1) and (2)

$$\delta(H_{N+1}(u) + \sum_{j=0}^{N} \lambda_j H_{N+1-j}(u)) = 0,$$
$$H_j(u) = l_j, \quad j = 0, \ldots, N \tag{18}$$

for some λ_j, and it minimizes $H_{N+1}(u)$ subject to the constraints (18). This gives an alternative characterization of N-band potentials. However, this formalism is not convenient for our purposes in the general case and we will only make use of it when $N = 1$.

3. The case of one-band solutions

Here we analyze the time behaviour of approximations to a cnoidal wave of the family (16). To this end, we consider semi-discrete (with discrete t and continuous x) one-step integrators for (1) of the form

$$U^{n+1} = \Phi_{\Delta t}(U^n), \quad n = 0, 1, \ldots, , \tag{19}$$

where Δt denotes the time step, $U^n = U^n(x)$ is a numerical solution at time level $t_n = n\Delta t$, $n = 0,1,\ldots$ and $\Phi_{\Delta t}$ approximates the Δt-flow $\phi_{\Delta t}$ of Eq. (1), written as an evolution equation in the corresponding phase space, described in (7). We first assume that, for all n, $U^n(x)$ is L-periodic, This has to be taken into account when, for practical purposes, the problem is discretized in space. Thus, if $U^0 = u_0(x - x_0)$ with u_0 a solution of (15), then the corresponding $U^n(x)$ approximates the solution $u = u(x, t, \gamma, \mu, x_0)$ of the form (16).

Our purpose here is to investigate the structure of the propagation of the errors $U^n(x) - u(x, t_n)$ for fixed x. The study of time behaviour of global error in the numerical approximation of ordinary differential equations may follow essentially two techniques. The first one considers asymptotic expansions of the global error [30,31,55,28,25]. The second one is the use of Backward Error Analysis. In the case of ordinary differential equations, this approach is based on the hypothesis of considering the numerical solution as the exact solution, at discrete times, of a modified differential equation. For a description of this theory, see e.g. [27,52] and many references therein (see also [26] for a comparison between the two techniques). For partial differential equations, however, this theory seems to be less expanded, see e.g. [18,44] for some special cases. In the present paper, asymptotic expansion of global error will be used.

Assuming the convergence of (19) for sufficiently small stepsizes Δt in the phase space H^1_{per}, we make some additional hypotheses.

(H1) The local error at u admits an expansion of the form

$$\Phi_{\Delta t}(u) - \varphi_{\Delta t}(u) = \Delta t^{r+1} l_{r+1}(u) + \Delta t^{r+2} l_{r+2}(u) + \cdots.$$

Here, $r \geqslant 1$ is the order of the method. We assume that the functions $l_{r+j}(u)$, $j = 1, 2, \ldots$, are smooth, independent of Δt and admit the group of translations in space as a symmetry group [16]. Last hypothesis can be obtained if the mapping $\Phi_{\Delta t}$ is invariant by this group, and is satisfied by most standard integrators, since the group is linear [56]. Note also that, since $l_{r+j}(u)$ is a function of u and derivatives, then it is L-periodic. Finally, the existence of the local error expansion can be obtained as a consequence of smoothness and consistency of the method [28].

(H2) The global error at u admits an asymptotic expansion of the form

$$U^n(x) - u(x, t_n, \gamma, \mu, x_0) = \sum_{j=0}^{r-1} \Delta t^{r+j} e_{r+j}(x, t_n) + \Delta t^{2r-1} z(x, t_n, \Delta t), \tag{20}$$

where

- $z(\cdot, t, \Delta t)$ is a smooth mapping such that for fixed t, $\|z(\cdot, t, \Delta t)\|_1 \to 0$ as $\Delta t \to 0$
- $e_{r+j}(x, t)$ are smooth error functions independent of Δt.

The existence of (20) is not restrictive. It can be deduced in some standard case, e. g. from the convergence and the hypothesis that the method admits an expansion of the form

$$\Phi_{\Delta t}(u) = u + \Delta t f(u) + \Delta t^2 d_2(u) + \Delta t^3 d_3(u) + \cdots,$$

where $f(u) = \mathbb{J}\delta H_2(u)$ and the functions $d_j(u)$ are analytic. The coefficient of Δt is $f(u)$ for consistent methods.

In (20), the remainder $z(\cdot, t, \Delta t)$, although goes to zero with the time step, does not have a uniform time behaviour. Therefore, this term reduces the time for which the propagation of the global error is explained by the error functions. However, since the expansion is up to order $O(\Delta t^{2r})$, this guarantees the dominance of the error terms e_{r+j}, with Δt sufficiently small, for long times, see [10].

Note that since we assume that $U^n(x)$ is L-periodic, from the equality

$$U^n(x+L) - u(x+L, t_n, \gamma, \mu, x_0) = U^n(x) - u(x, t_n, \gamma, \mu, x_0),$$

using (20), differentiating with respect to Δt and taking $\Delta t \to 0$, we have

$$e_{r+j}(x+L, t_n) = e_{r+j}(x, t_n), \quad j = 0, \ldots, r-1,$$

and hence

$$z(x+L, t_n, \Delta t) = z(x, t_n, \Delta t).$$

This implies that the error functions $e_{r+j}, j = 0, \ldots, r-1$ and the remainder z are periodic in space with period L. As well, the error functions $e_{r+j}, 0 \leqslant j \leqslant r-1$, satisfy nonhomogeneous versions of the linearized problem around u of the form [30,55,28]

$$\begin{aligned} &\partial_t e_{r+j} = \mathbb{J}H_2''(u)e_{r+j} + \sigma_j(u), \\ &e_{r+j}(x, 0) = 0. \end{aligned} \tag{21}$$

In (21), $H_2''(u)$ denotes the Hessian of the Hamiltonian at the cnoidal solution. It can be seen that for $0 \leqslant j \leqslant r-1$, the source term, $\sigma_j(u(x,t))$ in (21), only depends on $\mathbb{J}\delta H_2(u)$, the terms of the local error at u and their derivatives. This implies that $\sigma_j(u(\cdot, t))$ is L-periodic. However, these sources are hard to obtain. The simplest case, corresponding to $j = 0$, gives $\sigma_0 = -l_{r+1}$ and the problem for the leading term of the global error expansion

$$\begin{aligned} &\partial_t e_r = \mathbb{J}H_2''(u)e_r - l_{r+1}(u), \\ &e_r(x, 0) = 0. \end{aligned}$$

The analysis of the global error requires then to study the problems (21). Note that with the change of variables

$$X = x - \lambda t - x_0, \quad T = t,$$

the function $\mathbb{E}_{r+j}(X, T) = e_{r+j}(x, t)$ satisfies $(0 \leqslant j \leqslant r-1)$

$$\begin{aligned} &\partial_T \mathbb{E}_{r+j} = \mathbb{J}H''(u_0(X))\mathbb{E}_{r+j} + \sigma_j(u_0(X)), \\ &\mathbb{E}_{r+j}(X, 0) = 0 \end{aligned} \tag{22}$$

where $H(u) = H_2(u) + \lambda H_1(u)$. Therefore, the reference moving with the wave transforms the problems (21) into problems (22) with time independent coefficients. This makes the analysis of the error easier to be treated, but observe that this cannot be done in the general case of N-phase solutions, $N > 1$.

3.1. Analysis of linearized equations

We now search for the structure of the error functions e_{r+j} by analyzing the form of the solutions of (22). Since the steps of the study described below are common to all the equations (22), we try to investigate the form of the solution of a problem of the general type

$$\begin{aligned} &\partial_T \mathbb{E} = \mathbb{J}H''(u_0(X))\mathbb{E} + \sigma(u_0(X)), \\ &\mathbb{E}(X, 0) = 0 \end{aligned} \tag{23}$$

with $H(u) = H_2(u) + \lambda H_1(u)$. Then, the results about (23) will be applied to each error function.

We need to study some spectral properties of the operator

$$\mathbb{L} = \mathbb{J}H''(u_0), \quad H''(u_0) = -\partial_{XX} - 2u_0(X) + \lambda. \tag{24}$$

These are shown and explained in the following results. Hereafter, it will be understood that the derivatives are evaluated at the corresponding values of γ, μ, x_0 of u.

Theorem 1. *We consider the operator (24) defined in H^1_{per}. Then*

(i) *Zero is an eigenvalue of $\mathbb{L}$ with*

$$V_1 = \operatorname{Ker}_g \mathbb{L} \cap H^1_{\text{per},0} = \operatorname{Ker} \mathbb{L}^2 \cap H^1_{\text{per},0} = \operatorname{span}\left(\partial_X u_0, \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}\right).$$

Similarly, if $\mathbb{L}^*$ denotes the adjoint of $\mathbb{L}$, then

$$\operatorname{Ker}_g \mathbb{L}^* \cap H^1_{\text{per},0} = \operatorname{Ker}(\mathbb{L}^*)^2 \cap H^1_{\text{per},0} = \operatorname{span}(v(X), A(X))$$

with

$$v(X) = u_0(X) - \frac{\mu}{L}\mathbf{1}, \tag{25}$$

$$A(X) = P(X) - \frac{\langle P, \mathbf{1}\rangle}{L}\mathbf{1}, \quad P(X) = \int_0^L \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}\,\mathrm{d}y, \tag{26}$$

where $\mathbf{1}$ denotes the function that equals 1 for all X and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2[0, L]$.

(ii) *Furthermore, if*

$$V_2 = (\operatorname{Ker}_g \mathbb{L} \cap H^1_{\text{per},0})^{\perp} \cap H^1_{\text{per},0},$$

where the orthogonality is considered with respect to the usual inner product in $L^2[0, L]$ then

$$H^1_{\text{per}} = V_1 \oplus V_2 \oplus \operatorname{span}(\mathbf{1}). \tag{27}$$

Proof. This is done in several steps, where we will use some known results [5,12,13]. They are:

1. The operator $H''(u_0)$ is self-adjoint in H^1_{per} with
$$\operatorname{Ker} H''(u_0) = \operatorname{span}(\partial_X(u_0)). \tag{28}$$
2. We have the following properties:
$$\frac{\partial \lambda}{\partial \gamma} < 0, \tag{29}$$
$$H''(u_0)\mathbf{1} = 2u_0 - \lambda\mathbf{1}, \tag{30}$$
$$H''(u_0)\frac{\mathrm{d}u_0}{\mathrm{d}\gamma} = \frac{\partial \lambda}{\partial \gamma}u_0 + \frac{\partial \alpha}{\partial \gamma}\mathbf{1}. \tag{31}$$

Note first that if $\varphi \in H^1_{\text{per}}$, we can decompose

$$\varphi = \varphi - \frac{H_0(\varphi)}{L}\mathbf{1} + \frac{H_0(\varphi)}{L}\mathbf{1}$$

with $\varphi_1 = \varphi - \frac{H_0(\varphi)}{L}\mathbf{1} \in H^1_{\text{per},0}$. Since any function in $H^1_{\text{per},0}$ is, by definition, orthogonal to the function $\mathbf{1}$, then we have

$$H^1_{\text{per}} = H^1_{\text{per},0} \oplus \operatorname{span}(\mathbf{1}), \tag{32}$$

that will be used elsewhere.

First step of the proof is enclosed in the following:

Lemma 2

Under the conditions of Theorem 1,

$$\operatorname{Ker}\mathbb{L}\cap H^1_{\mathrm{per},0}=\operatorname{span}(\partial_X u_0),$$
$$\operatorname{Ker}\mathbb{L}^*\cap H^1_{\mathrm{per},0}=\operatorname{span}(v(X)),$$

with v given by (25).

Proof of Lemma 2. We will prove first equality, since the second one follows a similar argument. Take $\varphi\in H^1_{\mathrm{per},0}$ with $\mathbb{L}\varphi=0$. Then $H''(u_0)\varphi=C$ is constant. If $C=0$, then $\varphi\in Ker\ H''(u_0)$ and, following (28), $\varphi\in span(\partial_X u_0)$. Suppose that $C\neq 0$. Without loss of generality we may assume $C=1$. Then $H''(u_0)\varphi=\mathbf{1}$. Since u_0 is L-periodic, using (28) and the self-adjoint character of $H''(u_0)$, we have that the function $\mathbf{1}$ is orthogonal to the kernel of $H''(u_0)$, and therefore there is $\eta\in H^1_{\mathrm{per}}$ with $\mathbb{L}\eta=\mathbf{1}$. Suppose now that $\eta\in H^1_{\mathrm{per},0}$, that is $\langle\eta,\mathbf{1}\rangle=0$. Then, using self-adjointness and (30),

$$\langle H''(u_0)\eta,\mathbf{1}\rangle=\langle\eta,H''(u_0)\mathbf{1}\rangle=2\langle\eta,u_0\rangle-\lambda\langle\eta,\mathbf{1}\rangle=2\langle\eta,u_0\rangle.$$

And also

$$\langle H''(u_0)\eta,\mathbf{1}\rangle=\langle\mathbf{1},\mathbf{1}\rangle=L.$$

Therefore $\langle\eta,u_0\rangle=L/2$. On the other hand, since $\frac{du_0}{d\gamma}\in H^1_{\mathrm{per},0}$, we have $\langle H''(u_0)\eta,\frac{du_0}{d\gamma}\rangle=\langle\mathbf{1},\frac{du_0}{d\gamma}\rangle=0$. But also, using (31),

$$\langle H''(u_0)\eta,\frac{du_0}{d\gamma}\rangle=\langle\eta,H''(u_0)\frac{du_0}{d\gamma}\rangle=\frac{\partial\lambda}{\partial\gamma}\langle\eta,u_0\rangle+\frac{\partial\alpha}{\partial\gamma}\langle\eta,\mathbf{1}\rangle=\frac{\partial\lambda}{\partial\gamma}\langle\eta,u_0\rangle.$$

Then (29) implies $\langle\eta,u_0\rangle=0$, which is a contradiction. Thus η cannot be in $H^1_{\mathrm{per},0}$ and the problem $H''(u_0)\varphi=1$ does not have any solution in this space. This proves the first part of Lemma 2. □

We now prove first part of (i) in Theorem 1, since the second part about the adjoint can be proved in a similar way. Note that, according to (31),

$$\mathbb{L}\frac{du_0}{d\gamma}=\frac{\partial\lambda}{\partial\gamma}\partial_X u_0.$$

Then $\frac{du_0}{d\gamma}\in\operatorname{Ker}\mathbb{L}^2\cap H^1_{\mathrm{per},0}$. On the other hand, if $\varphi\in\operatorname{Ker}\mathbb{L}^2\cap H^1_{\mathrm{per},0}$, then $\mathbb{L}\varphi\in\operatorname{Ker}\mathbb{L}$. But

$$\langle\mathbb{L}\varphi,\mathbf{1}\rangle=\langle\varphi,\mathbb{L}^*\mathbf{1}\rangle=0.$$

Therefore $\mathbb{L}\varphi\in\operatorname{Ker}\mathbb{L}\cap H^1_{\mathrm{per},0}$. By applying Lemma 2, we have $\mathbb{L}\varphi=\alpha_1\partial_X u_0$, for some constant α_1; that is

$$H''(u_0)\varphi=\alpha_1 u_0+\alpha_2\mathbf{1},\tag{33}$$

for some other constant α_2. By using the identities (30) and (31), we can write the functions u_0 and $\mathbf{1}$ in terms of $H''(u_0)\frac{du_0}{d\gamma}$ and $H''(u_0)\mathbf{1}$ (see the remark at the end of the proof of Theorem 1). Translating this to (33), we can obtain some constants β_1,β_2 such that $\varphi-\beta_1\frac{du_0}{d\gamma}-\beta_2\mathbf{1}\in\operatorname{Ker}H''(u_0)$; hence

$$\varphi=\beta_1\frac{du_0}{d\gamma}+\beta_2\mathbf{1}+\beta_3\partial_X u_0,$$

for some other constant β_3. Since $\varphi\in H^1_{\mathrm{per},0}$, we have $\beta_2=0$ and therefore $\varphi\in\operatorname{span}(\partial_X u_0,\frac{du_0}{d\gamma})$. To complete the proof of the first part of (i) in Theorem 1, we have to see that the generalized kernel of $\mathbb{L}$ in $H^1_{\mathrm{per},0}$ coincides with $\operatorname{Ker}\mathbb{L}^2$. Take $\varphi\in\operatorname{Ker}\mathbb{L}^3\cap H^1_{\mathrm{per},0}$. Then, as before, $\mathbb{L}^2\varphi\in Ker\mathbb{L}$ and therefore it is of the form

$$\mathbb{L}\varphi=\beta_1\frac{du_0}{d\gamma}+\beta_2\partial_X u_0,$$

for some constants β_1,β_2. Taking the inner product with the function v given by (25) we have, on the one hand

$$\langle\mathbb{L}\varphi,v\rangle=\langle\varphi,\mathbb{L}^*v\rangle=0,$$

by using Lemma 2. On the other hand

$$\langle \mathbb{L}\varphi, v\rangle = \left\langle \beta_1 \frac{\mathrm{d}u_0}{\mathrm{d}\gamma} + \beta_2 \partial_X u_0, v \right\rangle = \beta_1 \left\langle \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}, u_0 \right\rangle.$$

And taking into account that

$$\left\langle \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}, u_0 \right\rangle = \frac{\mathrm{d}}{\mathrm{d}\gamma} H_1(u_0) = 1,$$

we have $\beta_1 = 0$ and $\mathbb{L}\varphi = \beta_2 \partial_X u_0 \in \operatorname{Ker}\mathbb{L} \cap H^1_{\mathrm{per},0}$. This implies $\varphi \in \operatorname{Ker}\mathbb{L}^2 \cap H^1_{\mathrm{per},0}$ and completes the proof. □

Now we attack part (ii) of Theorem 1. We have that (27) is a consequence of (32) and the following

Lemma 3. *With the notation of Theorem 1,*

$$H^1_{\mathrm{per},0} = V_1 \oplus V_2.$$

Proof of Lemma 3. It is not hard to see that $V_1 \cap V_2 = \{0\}$. Note also that, after some computations, if $w \in V_1$ then

$$w = \alpha_1 \partial_X u_0 + \alpha_2 \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}$$

with

$$\alpha_1 = -\langle w, A\rangle, \quad \alpha_2 = \langle w, v\rangle,$$

where v and A are given by (25) and (26), respectively and where we have used the property, proved before, that $\left\langle \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}, u_0 \right\rangle = 1$ (see proof of Theorem 1). Therefore any $\varphi \in H^1_{\mathrm{per},0}$ can be written as $\varphi = \varphi_1 + \varphi_2$ with

$$\varphi_1 = -\langle \varphi, A\rangle \partial_X u_0 + \langle \varphi, v\rangle \frac{\mathrm{d}u_0}{\mathrm{d}\gamma} \in V_1,$$

while $\varphi_2 = \varphi - \varphi_1$ satisfies $\varphi_2 \in H^1_{\mathrm{per},0}$ and

$$\langle \varphi_2, v\rangle = \langle \varphi_2, A\rangle = 0. \qquad \square$$

Remarks

(1) The function $\frac{\mathrm{d}u_0}{\mathrm{d}\mu} \in H^1_{\mathrm{per}}$ satisfies

$$H''(u_0)\frac{\mathrm{d}u_0}{\mathrm{d}\mu} = \frac{\partial\lambda}{\partial\mu} v - \frac{1}{L}\mathbf{1} \tag{34}$$

with $w = \frac{\mathrm{d}u_0}{\mathrm{d}\mu} - \frac{1}{L} \in H^1_{\mathrm{per},0}$. We can write (30) and (31) in the form

$$H''(u_0)\frac{\mathrm{d}u_0}{\mathrm{d}\gamma} = \frac{\partial\lambda}{\partial\gamma} v + \frac{2}{L}\mathbf{1}, \tag{35}$$

$$H''(u_0)\mathbf{1} = 2v + \left(\frac{2\mu}{L} - \lambda\right)\mathbf{1}. \tag{36}$$

Note finally that (34) implies that $w \in \operatorname{Ker}\mathbb{L}^2 \cap H^1_{\mathrm{per},0}$, so it is a combination of $\frac{\mathrm{d}u_0}{\mathrm{d}\gamma}$ and $\partial_X u_0$. Furthermore, it can be seen that two of the three Eqs. (34)–(36) are independent. These two results allow to write u_0 and $\mathbf{1}$ in terms of $H''(u_0)\frac{\mathrm{d}u_0}{\mathrm{d}\gamma}$ and $H''(u_0)\mathbf{1}$ in the proof of Theorem 1.

(2) The bases

$$\left\{\partial_X u_0, \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}\right\}, \quad \{A(X), v(X)\},$$

of V_1 and $W = \mathrm{Ker}_g \mathbb{L}^* \cap H^1_{\mathrm{per},0}$, respectively, are biorthogonal, since

$$\langle \partial_X u_0, v\rangle = \left\langle \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}, A\right\rangle = 0,$$
$$\langle \partial_X u_0, A\rangle = -\left\langle \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}, u_0\right\rangle = -\frac{\mathrm{d}}{\mathrm{d}\gamma} H_1(u_0) = -1,$$
$$\left\langle \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}, v\right\rangle = \left\langle \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}, u_0\right\rangle = \frac{\mathrm{d}}{\mathrm{d}\gamma} H_1(u_0) = 1.$$

We also remind that

$$\mathbb{L}\frac{\mathrm{d}u_0}{\mathrm{d}\gamma} = -\frac{\partial\lambda}{\partial\gamma}\partial_X u_0, \quad \mathbb{L}^* A = -\frac{\partial\lambda}{\partial\gamma} v - \frac{2}{L}\mathbf{1}.$$

We now apply these spectral results to the study of the structure of the solution of (23). According to Theorem 1, we decompose the source term $\sigma = \sigma_1 + \sigma_2 + \sigma_3$ with

$$\sigma_1 = \alpha_1 \partial_X u_0 + \alpha_2 \frac{\mathrm{d}u_0}{\mathrm{d}\gamma}, \quad \alpha_1 = -\langle \sigma, A\rangle, \alpha_2 = \langle \sigma, v\rangle,$$
$$\sigma_2 \in V_2, \quad \sigma_3 = \frac{\langle \sigma, \mathbf{1}\rangle}{L}\mathbf{1}.$$

Then the solution of (23) can be written as $\mathbb{E} = \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_3$ with $\mathbb{E}_j$ the solution of

$$\begin{aligned} &\partial_T \mathbb{E} = \mathbb{J} H''(u_0(X))\mathbb{E} + \sigma_j(u_0(X)),\\ &\mathbb{E}(X,0) = 0, \end{aligned} \tag{37}$$

We analyze the form of each component $\mathbb{E}_j$, $j = 1,2,3$. As far as $\mathbb{E}_1$ is concerned, we can apply spectral properties considered above and write

$$\mathbb{E}_1(T) = \alpha_1 T \partial_X u_0 + \alpha_2 \left(T\frac{\mathrm{d}u_0}{\mathrm{d}\gamma} - \frac{T^2}{2}\frac{\partial\lambda}{\partial\gamma}\partial_X u_0\right). \tag{38}$$

The form of $\mathbb{E}_3$ is not hard to obtain either;

$$\mathbb{E}_3(T) = \frac{\langle \sigma, \mathbf{1}\rangle}{L}\left(T - \frac{T^2}{2}\partial_X u_0\right). \tag{39}$$

Finally, the analysis of $\mathbb{E}_2$ requires first to study the solutions of the homogeneous version

$$\partial_T \mathbb{E} = \mathbb{L}\mathbb{E}, \tag{40}$$

with initial condition $\mathbb{E}(0)$ in V_2. Observe that the functional

$$Q(\varphi) = \langle H''(u_0)\varphi, \varphi\rangle, \tag{41}$$

is preserved by the solutions of (40), since this is a Hamiltonian system with $\frac{1}{2}Q(\varphi)$ as Hamiltonian. The idea here is to show that Q defines a norm in V_2 equivalent to the usual norm in H^1_{per}. As a consequence, we will see that the solutions of (40) with initial condition in V_2 are bounded in time.

Theorem 4. *If Q is the functional given by (41), then*

$$\alpha = \inf_{\varphi\in V_2, \|\varphi\|_0 = 1} Q(\varphi) > 0,$$

where $\|\cdot\|_0$ denotes the usual $L^2[0,L]$ norm.

Proof. Note first that, since we assume that u_0 is not an equilibrium of the Hamiltonian system, then $\lambda \neq 0$. We also may assume that $\lambda > 0$, since if $\lambda < 0$, we can always change the sign of the relative equilibrium problem and consider $-\lambda$ instead of λ, and the arguments would be similar, with the appropriate change of sign in $H(u_0)$. In physical terms, we are always considering a propagation to the right.

Some results in [12] implies

$$\inf_{\langle\varphi,\mathbf{1}\rangle=\langle\varphi,v\rangle=0,\|\varphi\|_0=1} Q(\varphi) \geqslant 0.$$

Therefore $\alpha \geqslant 0$. Let us assume that $\alpha = 0$ and then we will come to contradiction (see [57]). If $\alpha = 0$ then we have a sequence $\{\varphi_n\}, \varphi_n \in V_2$, $\|\varphi_n\|_0 = 1$ such that $Q(\varphi_n)\downarrow 0$. The explicit expression of $Q(\varphi)$ is

$$Q(\varphi) = \int_0^L (\lambda\varphi^2 - 2u_0\varphi^2 + (\partial_X\varphi)^2)\,\mathrm{d}X.$$

Note first that since $\|\varphi_n\|_0 = 1$, we have

$$\int_0^L (\varphi_n^2 + (\partial_X\varphi_n)^2)\,\mathrm{d}X \geqslant 1.$$

On the other hand, since $Q(\varphi_n)\downarrow 0$, for $0 < \epsilon < 1$ there is n_0 such that for $n \geqslant n_0$,

$$\int_0^L (\varphi_n^2 + (\partial_X\varphi_n)^2)\,\mathrm{d}X \leqslant \frac{C}{\lambda}\int_0^L 2u_0\varphi_n^2\,\mathrm{d}X + \epsilon,$$

for a constant $C > \max(\lambda,1)$ and therefore

$$1 \leqslant \int_0^L (\varphi_n^2 + (\partial_X\varphi_n)^2)\,\mathrm{d}X \leqslant \frac{C}{\lambda}\int_0^L 2u_0\varphi_n^2\,\mathrm{d}X + \epsilon. \tag{42}$$

This implies that $\|\varphi_n\|_1$ is uniformly bounded and therefore there is a subsequence, also denoted by $\{\varphi_n\}$ that converges weakly to some φ_* in the H^1_{per} norm. By weak convergence, $\varphi_* \in V_2$. Note also that in $H^1_{\mathrm{per},0}$, weak convergence implies uniform convergence [12]. As a consequence

$$\int_0^L u_0\varphi_n^2\,\mathrm{d}X \to \int_0^L u_0\varphi_*^2\,\mathrm{d}X, \tag{43}$$

as $n \to \infty$. By using (42) and (43) we have $\varphi_* \neq 0$. We now show that $Q(\varphi_*) = 0$. By Fatou's Lemma and the weak convergence,

$$||\varphi_*||_0 \leqslant \liminf ||\varphi_n||_0.$$

On the other hand, if $\|f\|_0 = 1$ [8],

$$\langle f, \partial_X\varphi_*\rangle \leqslant \liminf\langle f, \partial_X\varphi_n\rangle \leqslant \liminf ||\partial_X\varphi_n||_0.$$

Then

$$||\partial_X\varphi_*||_0 = sup_{||f||_0=1}\langle f, \partial_X\varphi_*\rangle \leqslant \liminf ||\partial_X\varphi_n||_0,$$

and with (43) we have

$$Q(\varphi_*) \leqslant \liminf Q(\varphi_n) = 0,$$

which, along with the fact that $\alpha \geqslant 0$, implies $Q(\varphi_*) = 0$.

Now, without loss of generality we can assume $\|\varphi_*\|_0 = 1$ (in other case, the division of φ_* by its norm does not affect to the arguments). We consider the functionals

$$J_1(\varphi) = \langle\varphi,\varphi\rangle, \quad J_2(\varphi) = \langle\varphi, v\rangle,$$
$$J_3(\varphi) = \langle\varphi, A\rangle, \quad J_4(\varphi) = \langle\varphi, \mathbf{1}\rangle.$$

The minimum of $Q(\varphi)$ in H^1_{per} with the constrains $J_1(\varphi) = 1, J_2(\varphi) = J_3(\varphi) = J_4(\varphi) = 0$, is attained in φ_*. Then there exist constants a_m, $m = 1(1)4$, such that

$$\delta Q(\varphi_*) = \sum_{m=1}^{4} a_m \delta J_m(\varphi_*).$$

By expanding the functionals, it can be seen that this implies

$$H''(u_0)\varphi_* = b_1\varphi_* + b_2 v + b_3 A + b_4 \mathbf{1},$$

for some constants b_m, $m = 1(1)4$. The inner products of $H''(u_0)\varphi_*$ with the functions $\mathbf{1}$, φ_*, $\partial_X u_0$, the use of (36), Theorem 1 and the facts that $\varphi_* \in V_2$ and that $Q(\varphi_*) = 0$ imply that $b_1 = b_3 = b_4 = 0$. Then $H''(u_0)\varphi_* = b_2 v$ and therefore $\mathbb{L}\varphi_* = b_2\partial_X u_0$. Thus, φ_* is of the form

$$\varphi_* = c_1\partial_X u_0 + c_2\frac{\mathrm{d}u_0}{\mathrm{d}\gamma},$$

for some constants c_1, c_2. But, since $\varphi_* \in V_2$, necessarily $c_1 = c_2$. Hence $\varphi_* = 0$, which is a contradiction. □

As a consequence, we have:

Theorem 5. *For any $\varphi \in V_2$, $\gamma > 0$, there are constants $c_1(\gamma)$, $c_2(\gamma) > 0$ such that*

$$c_1(\gamma)||\varphi||_1^2 \leqslant Q(\varphi) \leqslant c_2(\gamma)||\varphi||_1^2.$$

Proof. From the explicit expression of $Q(\varphi)$, it is clear that there is $c_2(\gamma) > 0$ such that $Q(\varphi) \leqslant c_2(\gamma)||\varphi||_1^2$ for any $\varphi \in V_2$. On the other hand, Theorem 4 proves that we can find $c(\gamma) > 0$ such that if $\varphi \in V_2$, $Q(\varphi) \geqslant c(\gamma)||\varphi||_0^2$. Finally, as in [12], this implies the existence of $c_1(\gamma) > 0$ such that if $\varphi \in V_2$, $Q(\varphi) \geqslant c(\gamma)||\varphi||_1^2$. □

We now apply these results to the solutions of the linearized Eq. (40) with initial condition in V_2. It is not hard to see that V_2 is invariant by $\mathbb{L}$. Therefore, if $\mathbb{E}(T)$ is the solution of (40) with $\mathbb{E}(0) \in V_2$, then $\mathbb{E}(T) \in V_2$ for all $T \geqslant 0$. As well, according to Theorem 5,

$$c_1(\gamma)||\mathbb{E}(T)||_1^2 \leqslant Q(\mathbb{E}(T)) = Q(\mathbb{E}(0)) \leqslant c_2(\gamma)||\mathbb{E}(0)||_1^2.$$

The conclusion is that solutions of (41) with initial condition in V_2 are bounded in time in the norm of H^1_{per}.

We show the influence of this result in the resolution of (37) with $j = 2$, that is with $\sigma_2 \in V_2$. A first consequence, via Duhamel's principle, is that $\mathbb{E}_2(T)$ grows, in the H^1_{per} norm, at most linearly with time. But in fact, $\mathbb{E}_2(T)$ is bounded in time. To see this, note that, since $\sigma_2 \in V_2$, we can define

$$s_2(X) = S_2(X) - \frac{\langle S_2, \mathbf{1}\rangle}{L}\mathbf{1}, \quad S_2(X) = \int_0^X \sigma_2(y)\,\mathrm{d}y.$$

Then $s_2 \in H^1_{\mathrm{per},0}$, $\partial_X s_2 = \sigma_2$ and, due again to the fact that $\sigma_2 \in V_2$, we have $\langle s_2, \partial_X u_0\rangle = -\langle \sigma_2, u_0\rangle = 0$. Hence, s_2 is in the range of $H''(u_0)$, so we can obtain $g \in H^1_{\mathrm{per}}$ with $H''(u_0)g = s_2$ and therefore $\mathbb{L}g = \sigma_2$. Furthermore, it is not hard to see that $g \in H^1_{\mathrm{per},0}$ (in fact, $\langle g, \mathbf{1}\rangle = \langle g, u_0\rangle = 0$). If we write

$$g = c_1\partial_X u_0 + c_2\frac{\mathrm{d}u_0}{\mathrm{d}\gamma} + g_2,$$

with constants c_1, c_2 and $g_2 \in V_2$, then

$$\sigma_2 = \mathbb{L}g = c_2\frac{\partial\lambda}{\partial\gamma}\partial_X u_0 + Lg_2 \in V_2.$$

Therefore $c_2 = 0$ and $g = c_1\partial_X u_0 + g_2$. Finally we can write

$$\mathbb{E}_2(T) = \int_0^T \mathrm{e}^{(T-\tau)\mathbb{L}}\sigma_2\,\mathrm{d}\tau = \mathrm{e}^{t\mathbb{L}}g - g,$$

where $\mathrm{e}^{t\mathbb{L}}g = c_1\partial_X u_0 + \mathrm{e}^{t\mathbb{L}}g_2$, with $\mathrm{e}^{t\mathbb{L}}g_2 \in V_2$. This proves that $\mathbb{E}_2(T)$ is bounded in time in the H^1_{per} norm.

Gathering the results (38) and (39) and this last one, we have the following:

Theorem 6. *The solution of (23) in H^1_{per} can be written in the form*

$$\mathbb{E}(T) = \alpha_1 T\partial_X u_0 + \alpha_2\left(T\frac{\mathrm{d}u_0}{\mathrm{d}\gamma} - \frac{T^2}{2}\partial_X u_0\right) + \alpha_3\left(T - \frac{T^2}{2}\partial_X u_0\right) + \mathbb{E}_2(T) \tag{44}$$

where

$$\alpha_1 = -\langle \sigma, A\rangle, \quad \alpha_2 = \langle \sigma, v\rangle, \quad \alpha_3 = \langle \sigma, \mathbf{1}\rangle,$$

and there is a constant $C(\gamma)$ *such that* $||\mathbb{E}_2(T)||_1 \leqslant C(\gamma)$ *for all* $T \geqslant 0$.

3.2. Application to the numerical solution

The structure of $\mathbb{E}(T)$ in (44) can be applied to each error function e_{r+j} of the asymptotic expansion of the global error (20). Reordering terms, we come to

Theorem 7. *Under the assumptions* (*H*1) *and* (*H*2) *we can write*

$$U^n(x) = u(x, t_n, \tilde{\gamma}, \tilde{\mu}, \tilde{x}_0) + \Delta t^r \widetilde{\mathbb{E}}(x, t_n) + \Delta t^{2r-1} \Gamma(x, t_n, \Delta t), \Delta t \to 0 \tag{45}$$

where

(i) *The new parameters are*

$$\begin{aligned}
\tilde{x}_0 &= x_0 + t_n \sum_{j=0}^{r-1} a_j \Delta t^{r+j} + \frac{t_n^2}{2} \sum_{j=0}^{r-1} \left(\langle \sigma_j(u_0), v\rangle + \frac{\langle \sigma_j(u_0), \mathbf{1}\rangle}{L} \right) \Delta t^{r+j}, \\
\tilde{\mu} &= \mu + t_n \left(\sum_{j=0}^{r-1} \langle \sigma_j(u_0), \mathbf{1}\rangle \Delta t^{r+j} \right), \\
\tilde{\gamma} &= \gamma + t_n \sum_{j=0}^{r-1} \left(\langle \sigma_j(u_0), v\rangle + b_j \langle \sigma_j(u_0), \mathbf{1}\rangle \right) \Delta t^{r+j},
\end{aligned} \tag{46}$$

for some constants a_j, $b_j, j = 0, \ldots, r-1$.

(ii) *The function* $\widetilde{E}$ *accumulates the terms* $E_{r+j,2}(t) \in V_2$ *of the error functions, in such a way that*

$$\widetilde{\mathbb{E}} = \sum_{j=0}^{r-1} \Delta t^j \mathbb{E}_{r+j,2}.$$

Consequently, $\widetilde{\mathbb{E}}$ *is bounded in time in the* H^1_{per} *norm.*

(iii) *The function* Γ *is a remainder that, for fixed* t,

$$||\Gamma(\cdot, t, \Delta t)||_1 \to 0, \quad \Delta t \to 0.$$

One of the points of (45) is the identification of some sources of growth with time of the error, specially those related to the parameters, that may have influence on the simulation. Formulas in (46) reflect the fact that the behaviour of the parameters through the simulation depends on the quantities $\langle \sigma_j(u_0), \mathbf{1}\rangle, \langle \sigma_j(u_0), u_0\rangle$, $j = 0, \ldots, r-1$, that is

$$\langle \sigma_j(u_0), \delta H_0(u_0)\rangle, \quad \langle \sigma_j(u_0), \delta H_1(u_0)\rangle, \quad j = 0, \ldots, r-1. \tag{47}$$

It can be shown [22] that if the numerical method preserves the quantities H_0 and H_1 up to order $2r$, that is, if

$$H_0(U^n) - H_0(U^0) = \mathrm{O}(\Delta t^{2r}), \quad H_1(U^n) - H_1(U^0) = \mathrm{O}(\Delta t^{2r}),$$

then the coefficients (47) vanish. In particular, this holds if the semi-discrete method preserves exactly those invariants.

We focus on the last two formulas of (46). The first one says that the mean value of the numerical solution deviates linearly in time from that of the theoretical solution, unless H_0 is an invariant of the scheme at least up to order $2r$. In the same way, the preservation of the numerical solution on the level set $\{H_1 = \gamma\}$ depends, at least at the degree of approximation given by formula (45), on the vanishing of (47); this can be obtained if the numerical integration preserves both quantities up to order $2r$. This modification of the original values μ and γ has some consequences in the simulation of the more physical parameters of the cnoidal wave. This is

treated in the following subsection. For instance, we can already say that the quadratic growth with time of the error in the parameter of the initial phase, first formula in (46), turns into a linear growth under the preservation conditions described above over the numerical integration.

3.3. Numerical experiments

Two purposes have this subsection. The first one is to illustrate the previous results, concerning the different error behaviour as a function of the preservation, up to certain order, of invariants of the equation; in this case those given by (4)–(6). The second one is to show the influence of these conservation properties of the time integrators on the simulation of some other relevant parameters of the wave, specially those with some physical meaning.

The description of the experiment is as follows: we consider a cnoidal wave solution of the family (14), with parameters $b_1 = 0.1$, $b_2 = 0.15$, $b_3 = 0.2$. This will be approximated by three time integrators of low order: the first one preserves the quantity H_0 but not H_1; the second one is designed to preserve H_1 but, on the contrary, it does not conserve H_0. Finally, the third one preserves both quantities. We avoid higher order of convergence because we think the experiments seem to be sufficiently illustrative and too much computational cost does not seem necessary.

For the space discretization of (1) and (2) we will consider a Fourier pseudospectral method. Due to the accuracy of the discretization, errors committed in space are negligible compared to the error due to the time integration. Then, time integration is developed in Fourier space. The three time integrators selected are the following: from the family of SDIRK methods

$$\begin{array}{c|cc} \nu & 0 \\ 1-2\nu & \nu \\ \hline & 1/2 & 1/2 \end{array}$$

we consider

- The method of order three, denoted by [M1], corresponding to the value $v = (3+\sqrt{3})/6$. It is an example of a method with preservation of H_0 [24] but not of H_1.
- A modification of the previous method, consisting of forcing the numerical solution to preserve H_1 by using projection techniques [27]. The resulting method, denoted by [M2], looses the preservation of H_0.
- The third method [M3] is designed to, from [M1], preserve the quantities (4) and (5). Here projection techniques are also used, but the projection is made involving both quantities.

Figs. 1–3 display, in a log–log scale, the discrete L^2 norm of the global error as a function of time for the methods [M1], [M2], [M3], respectively. The stepsizes are $\Delta t = 0.1, 0.05, 0.025$ and the final time $t = 10^4$. The three figures illustrate Theorem 7, concerning the influence of the invariant quantities on the growth with time of the error. The behaviour in the cases [M1] and [M2] is similar: the slope of the lines shows that the global error, in both cases, grows quadratically with time, although for [M2] this is observed from longer times ($t \approx 2000$ against $t \approx 600$ in the case of [M1]). This source of error growth may come from the behaviour of the parameter of the phase, as Theorem 7 suggests. However, following this theorem, the scheme [M3] generates a numerical solution that simulates the parameters μ and γ in a better way, while the error in the phase only grows linearly with time. This linear growth is observed in the global error behaviour, as Fig. 3 shows. Note that, due to the relative equilibrium condition (15), a time behaviour of the error similar to that of [M3] can be obtained with a method preserving two of the first three invariants of the problem.

As far as the simulation of the parameters of the wave involved is concerned, the methods exhibit different behaviour, as Theorem 7 suggests. We have already analyzed the phase parameter and now we focus on the following parameters: the speed, the maximum, the minimum and the amplitude of the wave. Figs. 4–6 show, again in a log–log scale, the error in time in the velocity of the wave for the methods [M1], [M2] and [M3],

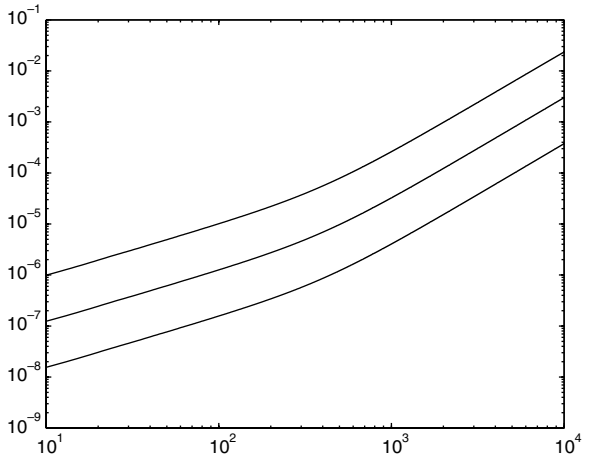

Fig. 1. Error vs time in log–log scale [M1]. The time steps are $\Delta t = 1/10, 1/20/1/40$.

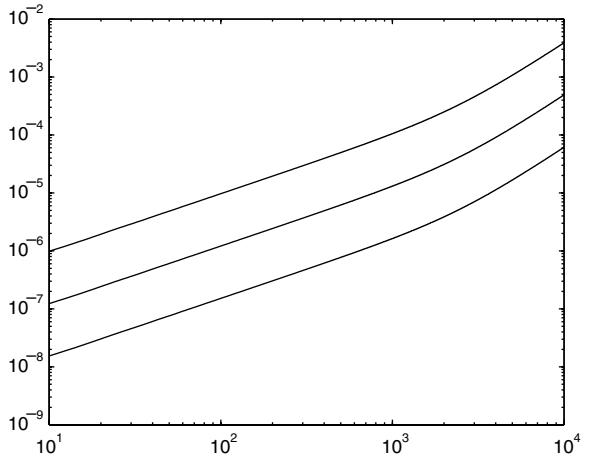

Fig. 2. Error vs time in log–log scale [M2]. The time steps are $\Delta t = 1/10, 1/20/1/40$.

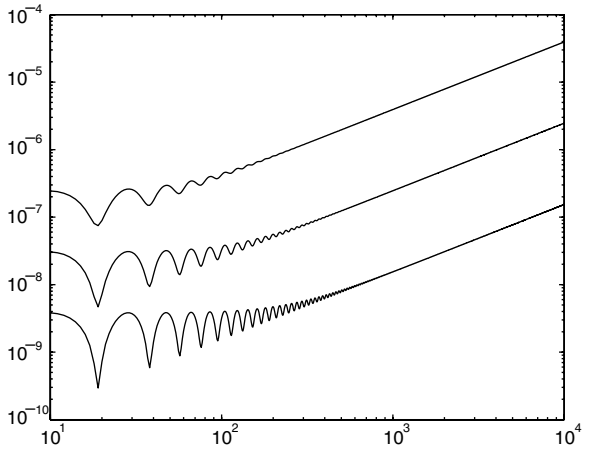

Fig. 3. Error vs time in log–log scale [M3]. The time steps are $\Delta t = 1/10, 1/20/1/40$.

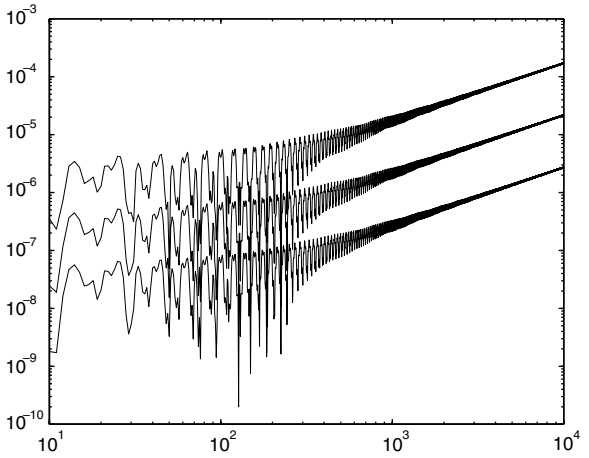

Fig. 4. Error in speed vs time in log–log scale [M1]. The time steps are $\Delta t = 1/10, 1/20/1/40$.

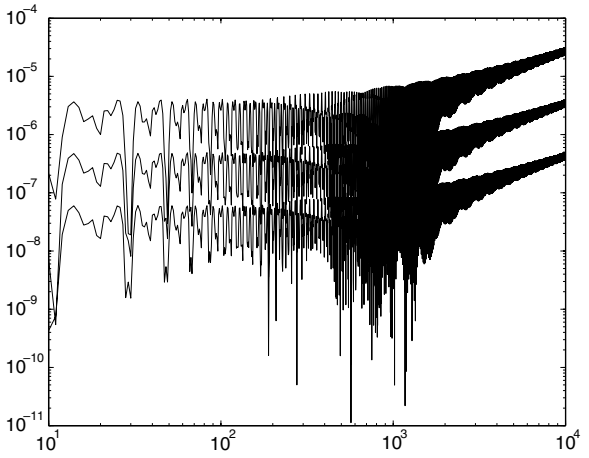

Fig. 5. Error in speed vs time in log–log scale [M2]. The time steps are $\Delta t = 1/10, 1/20/1/40$.

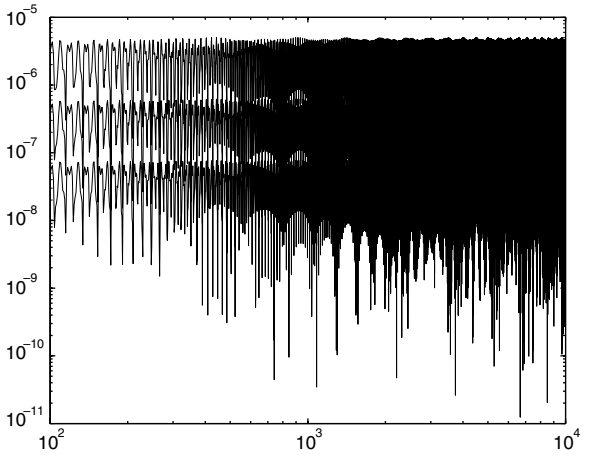

Fig. 6. Error in speed vs time in log–log scale [M3]. The time steps are $\Delta t = 1/10, 1/20/1/40$.

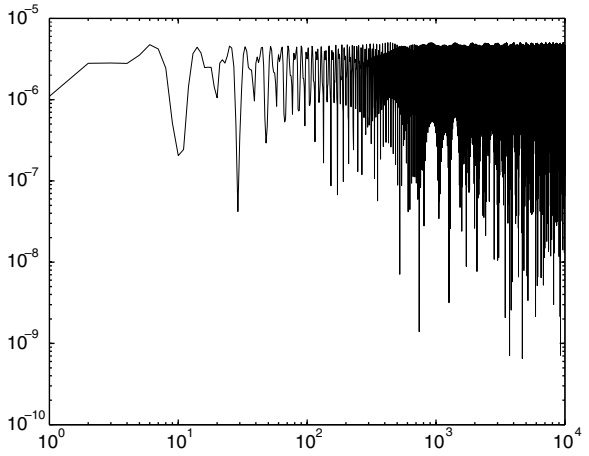

Fig. 7. Error in speed vs time in log–log scale [M3]. Fig. 6 with time step is $\Delta t = 1/10$.

respectively (Fig. 7 only contains the result in Fig. 6 corresponding to $\Delta t = 1/10$). The speed is measured in a standard way [6,7,19,21,15]. The steps sizes are $\Delta t = 0.1, 0.05, 0.025$. We observe a linear growth of the error for the 'nonconservative' methods [M1] and [M2], while for the third method, the error does not grow with time and it seems to provide a better simulation.

Some differences are also observed in the simulation of the minimum (b_2) and the maximum (b_3) of the wave. These are shown in Fig. 8 and 9, respectively. Fig. 8 displays the error in the maximum for the three methods against time; again [M1] with solid line, [M2] with dotted line and [M3] with dashed line. The maximum value seems to be better simulated in the case of [M3], at least during this time of integration. This method generates a numerical solution with a maximum that does not exhibit a linear time growth with respect to the real maximum, as in the case of the other two methods. Similar comments can be made for the case of the simulation of the minimum, shown in Fig. 9. The computation of these two parameters has also been carried out in an standard way, by using the pseudospectral discretization in space ([15] and references therein). The final parameter, the amplitude, is defined as the difference between the maximum and the minimum of the cnoidal wave. In our experiments, it has been measured in the same way and the results are shown in Fig. 10. Again, the advantages of the use of [M3] to simulate this parameter are also observed.

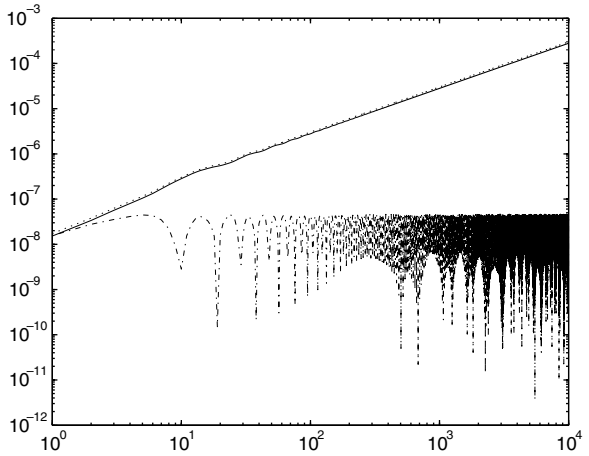

Fig. 8. Error in the maximum vs time in log–log scale. Solid line [M1], dotted line [M2] and dashed line [M3]. The time step is $\Delta t = 1/10$.

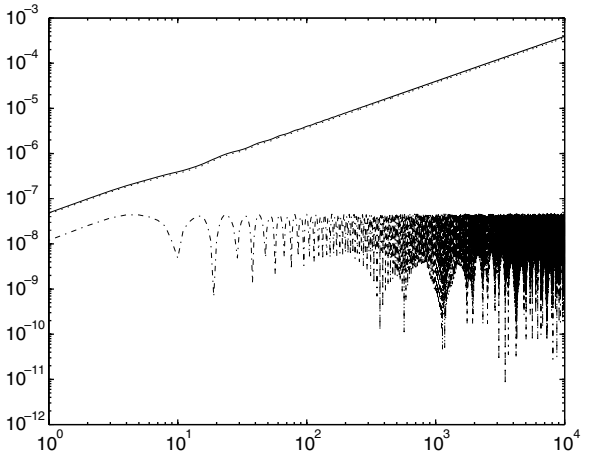

Fig. 9. Error in the minimum vs time in log–log scale. Solid line [M1], dotted line [M2] and dashed line [M3]. The time step is $\Delta t = 1/10$.

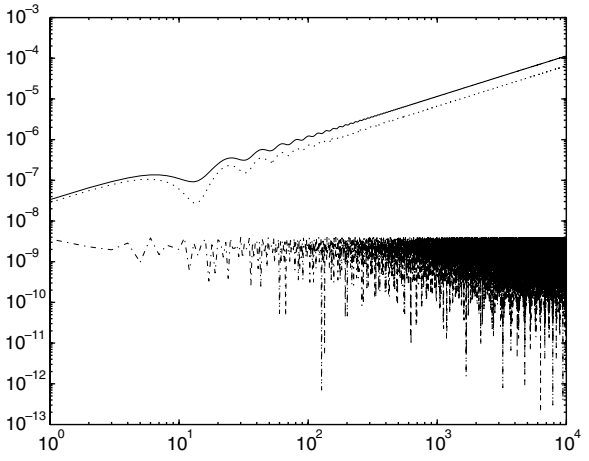

Fig. 10. Error in the amplitude vs time in log–log scale. Solid line [M1], dotted line [M2] and dashed line [M3]. The time step is $\Delta t = 1/10$.

4. Case of N-phase solutions, $N > 1$

Here we study the influence of the invariants of (1) and (2) in the simulation of general N-band solutions, with $N > 1$. An intuitive idea may be given by the decomposition of the solution (12) in the form [50]

$$u_N(x,t) = u_{\mathrm{cn}}(\vec{\eta}) + u_{\mathrm{int}}(\vec{\eta}), \tag{48}$$

where $\vec{\eta} = (\eta_1, \ldots, \eta_N)$, η_j given by (13). In (48), u_{cn} represents a superposition of cnoidal waves

$$u_{\mathrm{cn}}(\vec{\eta}) = \sum_{n=1}^{N} w_n(\eta_n), \tag{49}$$

with w_n a cnoidal wave of the form considered in the previous section and determined by η_n. The second term u_{int} in (48) represents the part of the solution corresponding to nonlinear interactions between the cnoidal components. In general, this cannot be considered small, in contrast to the case of N solitons [29]. However, the decomposition (48) and Section 3 suggest that the approximation to the solution u_N should be affected somehow by the conservative or nonconservative character of the time integrator considered; character not only with respect to the first three invariants, but also with respect to the first $N+1$ invariants. The analysis of this influence can be made by using similar arguments to those of the case of one-band solutions, treated

above. However, in the general case, the study of the linearized equation around an N-band solution must follow a different way, since now we cannot simplify the equations with a reference moving with the solution, as before. A first way of dealing with the problem could be looking for a decomposition of each error function of (20) with $u = u_N$,

$$e(x,t) = \sum_{n=1}^{N} e_n(\eta_n) + e_{\text{int}}(\vec{\eta}), \tag{50}$$

similar to (48) for the solution. Since, for each error function, the source term $\sigma(u_N(x,t))$ of the nonhomogeneous linearized KdV equation around u_N, Eq. (21) with $u = u_N$, depends on u_N and spatial derivatives, then it can also be written in a similar way

$$\sigma = \sum_{n=1}^{N} \sigma(w_n) + \sigma_{\text{int}}(\vec{\eta}), \tag{51}$$

where $\sigma(w_n)$ is the source term of the corresponding linearized KdV equation around $u = w_n$. Incorporating (50) and (51) into (21), we have (see [29])

(i) Each term e_n satisfies the nonhomogeneous linearized KdV equation around the simple cnoidal wave w_n with source term $\sigma(w_n)$.
(ii) The term e_{int} is a solution of the nonhomogeneous linearized KdV equation around $u = u_N$, with a source term involving u_{int}, w_n, e_n, $n = 1, \ldots, N$ and σ_{int}.

This way of analysis may serve to see more directly the influence of the previous study for cnoidal waves in a general N-phase solution simulation. However, it does not allow to provide a role to all the first $N+1$ invariants in the simulation. In this case, treating the nonhomogeneous Eq. (21) around $u = u_N$ directly may be more appropriate. In order to describe the solutions of this equation, we will make use of a basis of squared eigenfunctions of the corresponding eigenvalue problem of the Hill's operator associated to the solution of the KdV equation, mentioned in Section 2.

4.1. Linearized equations

As in the previous Section, in order to study the propagation of the error when approximating the N-phase solution u_N, we need to describe the solutions of nonhomogeneous linearized KdV equations around u_N,

$$\begin{aligned} &\partial_t e = \mathbb{J}H_2''(u_N(x,t))e + \sigma(u_N(x,t)), \\ &e(x,0) = 0 \end{aligned} \tag{52}$$

satisfied by the error functions in (20). We first consider the associated homogeneous equation

$$\partial_t e = \mathbb{J}H_2''(u_N(x,t))e = -e_{xxx} - 2(u_N e)_x. \tag{53}$$

The following result finds a set of solutions of (53) which forms a basis of H^1_{per} for each t. Its construction is made as a biorthogonal basis of a set of solutions of the adjoint linearized KdV equation around u_N,

$$v_t - v_{xxx} - 2u_N v_x = 0. \tag{54}$$

The basis with solutions of (54) consists of squared eigenfunctions of the main spectrum of (8). This result is given in [35] (see also [17,20,42,36,43]. We note that, in this case, u_N can be characterized by the parameters $E_{2k}, \gamma_k = \gamma_k^0, k = 1, \ldots, N$, described in Section 2. This is a different point of view from that of Section 3 for the case $N = 1$.

Theorem 8 [35]. *There is a set $\{\varphi_m(x,t)\}_{m\in\mathbb{Z}}$ of solutions of (53) with*

$$\varphi_{m_k} = \frac{\partial u_N}{\partial E_{2k}}, \quad \varphi_{-m_k} = \frac{\partial u_N}{\partial \gamma_k},$$

(for $k = 1, \ldots, N$ and where the derivatives are evaluated at the corresponding values E_{2k}, γ_k of u_N) that for each t forms a basis of H^1_{per}. There is also a set $\{\phi_m(x,t)\}_{m\in\mathbb{Z}}$ of solutions of (54), with ϕ_m written in terms of squared eigenfunctions of the main spectrum of (8), that for each t forms a basis of H^1_{per}. The function ϕ_{m_k} can be expressed in terms of squared eigenfunctions associated exclusively to E_{2k}, $k = 1, \ldots, N$ and $\phi_0 = 1$. Furthermore, the bases $\{\varphi_m(x,t)\}_{m\in\mathbb{Z}}$ and $\{\phi_m(x,t)\}_{m\in\mathbb{Z}}$ are biorthogonal with respect to the usual $L^2[0,T]$ inner product.

Note that if, for fixed t, $\psi(\cdot,t) \in H^1_{\mathrm{per}}$, we can write

$$\psi = \sum_{m\in\mathbb{Z}} \psi_m(t)\varphi_m, \quad \psi_m(t) = \frac{\langle \psi, \phi_m\rangle}{||\phi_m||_0^2}.$$

In particular, for $k = 1, \ldots, N$,

$$\psi_{m_k}(t) = \frac{\langle \psi, \phi_{m_k}\rangle}{||\phi_{m_k}||_0^2}.$$

Since ϕ_{m_k} can be written only in terms of the squared eigenfunctions associated to E_{2k}, then [42,34], each of the variational derivatives $\delta H_j(u_N), j = 0, \ldots, N$ can be expressed as a combination of the ϕ_{m_k} in such a way that

$$\langle \psi, \phi_0\rangle = \langle \psi, \phi_{m_k}\rangle = 0, \quad k = 1, \ldots, N,$$

if and only if

$$\langle \psi, \delta H_j(u_N)\rangle = 0, \quad j = 0, \ldots, N.$$

In particular, in the case $m = 0$, we have

$$\psi_0(t) = \frac{\langle \psi, 1\rangle}{L} = \frac{\langle \psi, \delta H_0(u_N)\rangle}{L}.$$

This may have a consequence in the behaviour of numerical approximations to u_N. By using Duhamel's principle, the solution of (52) can be written as

$$e(x,t) = \int_0^t T(t,s)\sigma(u_N(x,s))\,\mathrm{d}s. \tag{55}$$

In (55), $(t,s) \mapsto T(t,s)$ is the operator of propagation for the linearized KdV (53); this means that $T(t,s)f(\cdot,s)$ is the solution at t of (53) with initial condition $f(\cdot,s)$ at $t = s$. By using the previous results, the operator T can be decomposed into the form

$$T(t,s) = T_1(t,s) + T_2(t,s)$$

with

$$T_1(t,s)\psi(x,s) = D_0(s)\varphi_0(x,t) + \sum_{k=1}^{N} D_k(s)\varphi_{m_k}(x,t) + L_k(s)\varphi_{-m_k}(x,t),$$

$$D_0(s) = \frac{\langle \psi, \phi_0\rangle}{L}, D_k(s) = \frac{\langle \psi, \phi_{m_k}\rangle}{||\phi_{m_k}||_0^2}, \quad k = 1, \ldots, N,$$

$$L_k(s) = \frac{\langle \psi, \phi_{-m_k}\rangle}{||\phi_{-m_k}||_0^2}, \quad k = 1, \ldots, N,$$

$$T_1(t,s)\psi(x,s) = \sum_{m\in\mathbb{Z}, m\neq\pm m_k} \frac{\langle \psi, \phi_m\rangle}{||\phi_m||_0^2}\varphi_m(x,t).$$

Then the solution e in (55) can be written as

$$e(x,t) = \mathrm{e}^{[1]}(x,t) + \mathrm{e}^{[2]}(x,t)$$

with

$$e^{[1]}(x,t) = \left(\int_0^t D_0(s)\,ds\right)\varphi_0(x,t) + \sum_{k=1}^{N}\left(\int_0^t D_k(s)\,ds\right)\varphi_{m_k}(x,t) + \left(\int_0^t L_k(s)\,ds\right)\varphi_{-m_k}(x,t), \tag{56}$$

$$D_k(s) = \frac{\langle \sigma(u_N(\cdot,s)), \phi_{m_k}\rangle}{||\phi_{m_k}||_0^2}, \quad k = 1,\ldots,N, D_0(s) = \frac{\langle \sigma(u_N(\cdot,s)), \phi_0\rangle}{L}, \tag{57}$$

$$L_k(s) = \frac{\langle \sigma(u_N(\cdot,s)), \phi_{-m_k}\rangle}{||\phi_{-m_k}||_0^2}, \quad k = 1,\ldots,N. \tag{58}$$

$$e^{[2]}(x,t) = \int_0^t T_2(t,s)\sigma(u_N(x,s))\,ds. \tag{59}$$

The application of (56)–(59) to each error function of the global error expansion (20) with respect to u_N allows to obtain an asymptotic expression for the numerical solution of the form

$$U^n(x) = u_N(x,t_n,\tilde{E}_{2k},\tilde{\gamma}_k,\tilde{\mu}) + \Delta t^r F(x,t_n) + \Delta^{2r-1} Z(x,t_n,\Delta t). \tag{60}$$

This is the version of formula (45) for the N-phase solution u_N. It deserves some comments.

(i) The numerical solution incorporates a modified N-phase wave with new parameters

$$\widetilde{E}_{2k} = E_{2k} + \sum_{j=0}^{r-1}\Delta t^{r+j}\left(\int_0^t D_k^{[r+j]}(s)\,ds\right), \quad k = 1,\ldots,N,$$

$$\tilde{\gamma}_k = \gamma_k + \sum_{j=0}^{r-1}\Delta t^{r+j}\left(\int_0^t L_k^{[r+j]}(s)\,ds\right), \quad k = 1,\ldots,N,$$

where $D_k^{[r+j]},\ k = 1,\ldots,N; L_k^{[r+j]},\ k = 1,\ldots,N$ are of the form (57) and (58) with $\sigma = \sigma_j$ as a source term, see (21), indicated by the superscript. Note that the behaviour in time of the integrals

$$\int_0^t D_k^{[r+j]}(s)\,ds, \quad \int_0^t L_k^{[r+j]}(s)\,ds,$$

depends on the time behaviour of the functions $\phi_{\pm m_k}$. Some previous comments show that if the method satisfies the conditions

$$\langle \sigma_j(u_N(x,t)), \delta H_k(u_N(x,t))\rangle = 0, \quad t \geqslant 0, k = 0,\ldots,N,\ \ j = 0,\ldots,r-1,$$

then $D_k = 0, k = 1,\ldots,N$ and the numerical solution provides a better simulation of the parameters E_{2k}, $k = 1,\ldots,N$. In particular, this holds if the method preserves, at least up to certain order, the first $N+1$ invariants of (1) and (2).

(ii) The second term $F(x,t)$ includes the parts $e_{r+j}^{[2]}(x,t)$ of the error functions. Again, its behaviour in time depends on the behaviour of the functions $\phi_m(x,t), m \neq m_k$.

(iii) The third term is a remainder that, for fixed t, goes to zero as $\Delta t \to 0$, with a not uniform time behaviour.

The better simulation of the parameters E_{2k} provided by the method with conservation properties can be positively felt in the simulation of the parameters in (12), due to their dependence on the E_{2k}. Note that, if we consider the decomposition (48) of the solution u_N, the numerical behaviour suggested by (60) affects to both terms, since the new parameters $\widetilde{E}_{2k}, \tilde{\gamma}_k$ must appear in both the cnoidal components (49) and the interaction part. The good properties of the method with respect to the invariant quantities of the problem result in a better simulation of the physical parameters of the wave, including the mean value, wherever they act, in the cnoidal part or in the interactions.

Acknowledgment

This work was supported by the Greece–Spain R&D Collaboration project Acción Integrada HG2004-0010.

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